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On open superstring partition function in inhomogeneous rolling tachyon background

Angelos Fotopoulos[ab] **and Arkady A. Tseytlin**[ca*]

[a] *Theoretical Physics Group, Blackett Laboratory, Imperial College London SW7 2BZ, U.K.*
[b] *Centre de Physique Theorique, Ecole Polytechnique 91128 Palaiseau, France*
[c] *Smith Laboratory, The Ohio State University Columbus OH 43210, U.S.A.*
E-mail: Angelos.Fotopoulos@cpht.polytechnique.fr, tseytlin@mps.ohio-state.edu

Abstract: We consider open superstring partition function Z on the disc in time-dependent tachyon background $T = f(x_i)e^{\mu x_0}$ where the profile function f depends on spatial coordinates. We compute Z to second order in derivatives of f and compare the result with some previously suggested effective actions depending only on first derivatives of the tachyon field. We also compute the target-space stress-energy tensor in this background and demonstrate its conservation for the linear profile $f = f_0 + q_i x^i$ corresponding to a marginal perturbation. We comment on the role of the rolling tachyon with the linear spatial profile in the decay of an unstable D-brane.

Keywords: D-branes, Conformal Field Models in String Theory, Tachyon Condensation.

*Also at Lebedev Physics Institute, Moscow.

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1. Introduction

Understanding tachyon condensation and possible role and meaning of tachyon effective action in string theory is an old and important problem. In general, trying to find an effective action for tachyon field only does not seem to make much sense since the scale of masses of an infinite set of massive string modes is the same as that of the tachyon mass, and thus keeping the tachyon while integrating out all other string modes may look unjustified. One may hope, however, that in certain situations (like in much discussed examples of non-BPS D-brane decay or brane–anti-brane annihilation) some aspects of string dynamics can be captured by an effective field-theory action involving only tachyon field (and massless modes): all other massive string modes may effectively decouple at a vicinity of certain conformal points. Reliable information about open string tachyon effective actions should be important, in particular, for current attempts of cosmological applications of string theory.

One important message of studies of open string tachyon condensation is that the form of tachyon effective action may depend on a choice of region in field space where it should be valid. For example, near the standard perturbative string vacuum $T = 0$ one may try to reconstruct tachyon effective action from string S-matrix by assuming that the tachyon is the only asymptotic state and by formally expanding the string scattering amplitudes in powers of momenta (assuming some off shell continuation). One then gets (we use signature $-+, \dots, +$, $m = 0, 1, 2, \dots$, and set $\alpha' = 1$)

$$L = -\frac{1}{2}(\partial_m T)^2 + \frac{1}{2}\mu^2 T^2 - g_1 T^4 + \cdots , \tag{1.1}$$

where $m_{tach}^2 = -\mu^2$. The linear part of the equations of motion following from (1.1) is the same as the leading-order tachyon beta-function

$$\partial_m^2 T + \mu^2 T = 0 , \qquad \mu_{\rm bose}^2 = 1 , \qquad \mu_{\rm super}^2 = \frac{1}{2} . \tag{1.2}$$

Since the action (1.1) is reconstructed from tachyon S-matrix near the tachyon vacuum, the applicability of local derivative expansion is doubtful: its form depends on a particular ad hoc assumption (not apparently encoded in the tachyon S-matrix) about off-shell continuation. A separation into derivative-independent and derivative-dependent terms is ambiguous; in particular, the coefficient of the derivative-independent T^4 term can be changed by a local field redefinition.

One may hope to do better by expanding near an end-point [1, 2] of tachyon evolution ($T \gg 1$) where tachyon gets “frozen” and one may expect that derivative expansion may make more sense. Then, following [3, 4, 5], one finds from the derivative expansion of the superstring partition function on the disc [6]

$$L = -e^{-\frac{1}{4}T^2} \left[1 + \frac{1}{2}(1 + b_1 T^2)(\partial_m T)^2 + \cdots \right] , \tag{1.3}$$

where dots stand for higher-derivative terms and $b_1 = \ln 2 - \frac{1}{2}$.

Here we included $\partial^2 T$ contribution and then integrated it by parts. The field redefinition ambiguity (the cutoff dependent coefficient of $\partial^2 T$ term) was fixed so that the linear part of the resulting effective equations agrees [6] with the linear part of the tachyon beta-function (1.2), i.e. like (1.1) this action reproduces correctly the value of the tachyon mass near $T = 0$. One may hope that this action may be used to interpolate between the regions of small T (vicinity of the tachyon vacuum) and large T (vicinity of a new vacuum where tachyon and other open string modes get frozen).

Extending the action (1.3) to higher orders in derivative expansion appears to be a complicated task. It is not clear if it makes sense to sum all terms in derivative expansion that depend only on the first but not on higher derivatives of T, e.g., by evaluating the string partition function on $T = q_m x^m$ background [5], since for finite q_m the linear tachyon background is not a solution of the resulting equations of motion. In general, it is not clear how to interpret a first-derivative tachyon lagrangian

$$L = -V(T)\,\mathrm{K}(\partial T)\,, \tag{1.4}$$

or, in particular, an often-discussed "tachyon DBI" lagrangian [7, 8, 9] (see also [10], cf. [11])

$$L_{\rm TDBI} = -V(T)\sqrt{-\det(\eta_{mn} + \partial_m T \partial_n T)} = -V(T)\sqrt{1+(\partial_m T)^2}\,. \tag{1.5}$$

Indeed, there is no a priori reason to expect that higher-derivative terms omitted in (1.4) should be small on solutions of the resulting equations, unless $V(T) \approx$const (which is not the case at least near $T = 0$ where one should get the tachyon mass term).[1] This is to be contrasted to the usual relativistic particle action or DBI action where higher-derivative "acceleration" terms can be indeed consistently ignored since a constant velocity motion or $F_{mn} =$ const is always a solution.

Alternatively, one may try also to reconstruct the tachyon effective action at a vicinity of other exact conformal points, e.g., time-dependent background which should represent an exact boundary conformal theory [13, 14] (Minkowski version of euclidean CFT of [15])

$$T = f_0 e^{\mu x^0} + \tilde{f}_0 e^{-\mu x^0}\,. \tag{1.6}$$

Its special case is the "rolling tachyon" background [13, 14]

$$T = f_0 e^{\mu x^0}\,. \tag{1.7}$$

The disc (super)string partition function in this background(1.7) was recently computed in [16], suggesting that the corresponding "potential" term should look like

$$V_0(T) = \frac{1}{1+\frac{1}{2}T^2}\,. \tag{1.8}$$

A remarkable observation made in a subsequent paper [17] is that demanding that a generic first-derivative lagrangian (1.4) should have (1.6) (with $\mu = 1/\sqrt{2}$ in the superstring case) as its exact solution fixes its time-derivative part[2] to be

$$L = -\frac{1}{1+\frac{1}{2}T^2}\sqrt{1+\frac{1}{2}T^2 - (\partial_0 T)^2}\,. \tag{1.9}$$

If we assume that (1.9) has a direct Lorentz-covariant generalization we are led to

$$L_{\rm KN} = -\frac{1}{1+\frac{1}{2}T^2}\sqrt{1+\frac{1}{2}T^2 + (\partial_m T)^2} = -\frac{1}{2}(\partial_m T)^2 + \frac{1}{4}T^2 + \cdots\,. \tag{1.10}$$

[1]For a discussion of various first-derivative tachyon actions see [12].

[2]Demanding only that (1.7) is a solution does not fix the action uniquely (see also [12], where the suggestion to fix the form of the first-derivative tachyon action by requiring that it admits an exactly marginal static tachyon background $T = a\sin\frac{x}{\sqrt{2}}$ as its exact solution was made). For a discussion of a bosonic string variant of the argument of [17] see [18].

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The action (1.10) does agree with (1.8) when evaluated on the background (1.7) or (1.6)[3] and, after a field redefinition [17], $\frac{T}{\sqrt{2}} \to \sinh \frac{\tilde{T}}{\sqrt{2}}$, becomes

$$L = -\tilde{V}(\tilde{T})\sqrt{1 + (\partial_m \tilde{T})^2}\,, \qquad \tilde{V} = \frac{1}{\cosh \frac{\tilde{T}}{\sqrt{2}}}\,. \tag{1.11}$$

This seems to vindicate the TDBI action (1.5) (discussed, e.g., in [19, 8]), but there are several questions remaining.

One is the range of validity of the action (1.10). As was proposed in [17], this action should be valid for tachyon fields which are "close" to the exactly marginal background (1.7), i.e. for $T = f(x^m)e^{\mu x^0}$ where f is a slowly changing function. The idea of [17] was to choose a particular direction in space-time, for which the exactly marginal background is $T = f_0 e^{\mu x^0}$ and then to expand in small *spatial* momenta near this point. While this prescription may seem not to be Lorentz-covariant, one expects that this breaking of Lorentz invariance is "spontaneous", i.e. the corresponding effective action summarizing dynamics of small perturbations near the exact conformal point can still be chosen Lorentz-invariant.

Indeed, in addition to the argument in favor of (1.9) based on having (1.6) as an exact solution and having agreement with (1.8), ref. [17] contained also an apparently independent S-matrix based argument supporting the existence of an action with first derivatives only that reproduces the leading (quadratic in spatial momenta) terms in the corresponding string amplitudes computed using an analytic continuation from euclidean space expressions.[4]

The relation of this second argument (assuming momentum conservation in all directions in euclidean space) to the first one referring to the partition function (1.8) where one does not integrate over the zero mode of x^0 and thus does not impose momentum conservation in x^0 direction is not obvious at the moment and may be quite subtle. Also, it is not clear a priori (independently of the first argument referring to having (1.6) as an exact solution) why to reproduce the leading $k_i k_j$ terms in the n-point tachyon scattering amplitudes computed using the euclidean continuation prescription of (1.10) one needs an action involving all powers of derivatives of T: while all powers of $\partial_m T$ do contribute to the single independent coefficient (1.10) of the quadratic spatial momentum term in each n-point amplitude, that term may well be reproduced just by the lagrangian $L = U_0(T) + U_1(T)(\partial_m T)^2$. Here U_0 and U_1 are power series in T such that the corresponding field-theory amplitude matches the leading term in

[3] At small $\tilde{T}$, the leading quadratic terms in this action match the first two terms in (1.1), i.e. like (1.3) it reproduces the correct value of the tachyon mass near $T = 0$. However, the leading terms in derivative expansion in (1.5) are not related to (1.3) by a field redefinition (which may be attributed to their different ranges of validity, cf. [17]).

[4] One expects that the effective action should be reproducing scattering amplitudes with very special kinematics where all tachyons have small spatial momenta, i.e. moving very slowly. It is also assumed that one first expands the string (and field theory) amplitudes in spatial momenta and then imposes momentum conservation.

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the one-shell string theory amplitude $\mathcal{A}_n \sim c_n k_i k_j + \cdots$. The preference of (1.10) may then be attributed to a specific scheme choice allowing to have (1.6) as an exact solution.[5]

Given somewhat indirect nature of the above arguments it would obviously be interesting to support them by explicit scattering amplitude computations and also to establish a precise relation between the euclidean continuation prescription used in the string S-matrix considerations and the real-time partition function computation [16] leading to (1.8). This was part of our original motivation in the present paper.[6]

In particular, it would be important to see if one can reproduce the TDBI action (1.10) *directly* from the *string path integral*, just like one can obtain the BI action $L(F)$ [20] and derivative (∂F) corrections to it [21] by expanding near the conformal point represented by a constant abelian gauge field strength background.

In general, the *exact* expression for the string partition function evaluated on a background $(x^m = (x^0, x^i))$

$$T(x^0, x^i) = f(x^0, x^i) e^{\mu x^0} \tag{1.12}$$

should indeed be Lorentz-covariant, but that need not apply to its first-derivative part only. Note also that the leading-order condition of marginality of such background, i.e. the equation $\partial_m^2 T + \frac{1}{2} T = 0$ $(\mu = \frac{1}{\sqrt{2}})$ becomes

$$\partial_i^2 f - \partial_0^2 f - \sqrt{2} \partial_0 f = 0 \,, \tag{1.13}$$

i.e. it mixes first and second derivatives. More precisely, for marginal perturbation like $f = f_0 e^{-\nu x^0 + ik\cdot x}$ we have $\nu = \frac{1}{\sqrt{2}} k^2 + O(k^4)$, so expanding in k_i or in spatial derivatives we have $\partial_0 f \sim \partial_i^2 f \sim (\partial_i f)^2$. It is here that the existence of a specific scheme choice [17] should be important, and one would like to understand how this scheme should be defined in the context of a standard (real-time) perturbative expansion of the string partition function.

One may notice that the action (1.10) does not admit

$$T = (f_0 + q_i x^i) e^{\frac{x^0}{\sqrt{2}}} \tag{1.14}$$

[5] Assuming, as suggested in [17], that (1.10) applies at the vicinity of the rolling tachyon background (1.7), it may not be a priori clear why the first-derivative action like (1.10) should be a useful tool: all higher-derivative terms ignored in (1.10) are of the same order on this exponential background (and they are small only near $x_0 = 0$ or small values of T where one can in any case ignore the non-linear terms in (1.10)). In general, higher-derivative terms (that should again admit (1.6) as an exact solution) may be crucial for correctly reproducing string dynamics (like string fluctuation spectrum) at a vicinity of the rolling tachyon background. However, as suggested in [17], there should exist a scheme in which all higher-derivative terms can be effectively traded for the first-derivative ones.

[6] Trying to put (1.10) as opposed to (1.9) on a firmer footing is important also since most of the previous discussions of the tachyon decay were in the homogeneous (space-independent) tachyon case, and while they indeed agree [8] with the lagrangian (1.9) depending only on $\partial_0 T$, the range of validity and practical utility of its inhomogeneous version (1.10) does not seem to be well understood at the moment.

as its exact solution, while this "nearby" background which solves (1.13) is expected, as suggested by the finiteness of the corresponding partition function discussed below, to be an exactly marginal perturbation.[7]

This does not, however, imply a contradiction but rather that it may be that (1.14) is an exact solution in a different scheme than the one implied in [17] in which (1.10) is supposed to be valid.[8] Indeed, we may consider (1.14) as a linear in q_i approximation to an exactly marginal background which is simply a boost of (1.7) ($q_m q^m = -\frac{1}{2}$)

$$\begin{aligned} T &= e^{q_m x^m} = e^{\sqrt{\frac{1}{2}+q_i^2}x^0 + q_i x_i} \\ &= \left[1 + q_i x_i + \frac{1}{2}q_i q_j x_i x_j + \frac{1}{\sqrt{2}}q_i^2 x^0 + O(q^3)\right] e^{\frac{x^0}{\sqrt{2}}} , \end{aligned} \tag{1.15}$$

and which is thus an exact solution of (1.10). The two backgrounds (1.14) and (1.15) may then be related by a field redefinition reflecting change of schemes in which each of these backgrounds is an exact solution. Notice that this field redefinition should necessarily involve time derivatives since it should transform (1.14) into a particular case of (1.12).[9]

With the motivation to try to understand better the structure of the tachyon effective action in the vicinity of the rolling background (1.7), and, eventually, a scheme choice in which (1.10) should be valid, we would like to compute the leading terms in derivative expansion of the string partition function on the disc

for the inhomogeneous tachyon background (1.12) that generalizes (1.7). We shall mostly consider the case when the profile function f in (1.12) depends only on the *spatial* coordinates, i.e. $f = f(x_i)$. Such background is marginal (i.e. satisfies (1.2)) if $\partial_i^2 f = 0$.

As a result, the expansion of the string partition function Z in derivatives of f seems as well-defined as the expansion in derivatives of any massless-level scalar mode.

Unfortunately, as we shall see below, expanding Z near f=const it does not appear to be straightforward to sum up all terms depending only on the *first* spatial derivative $\partial_i T = \partial_i f\, e^{\mu x^0}$ of the tachyon. In general, separation of the terms that depend only on ∂T and not on higher derivatives is *ambiguous* (in particular, in view of the presence of an overall potential factor and a possibility to integrate by parts).[10]

[7]By "exactly marginal" here we mean only that it solves the beta-function equations to all orders. While a non-normalizable nature of this background may be problematic for a CFT interpretation, such T is a natural counterpart of a linear vector potential field describing a gauge field with a constant strength, i.e. $A = A_m \dot{x}^m = \frac{1}{2}F_{mn}x^m\dot{x}^n$. Like a constant gauge field it cannot be regularly expanded in plane waves.

[8]We are grateful to D. Kutasov for an explanation that follows.

[9]An interesting question is if such a field redefinition is not changing physics: while (1.15) does not describe a D-brane at $x^0 \to \infty$, the background (1.14) may be thought of describing a co-dimension one D-brane, see below.

[10]Moreover, one is used to think that if Z or the effective action is computed in derivative expansion in $\partial_i T \sim \partial_i f$, one should treat all terms with the same number of derivatives on an equal footing, e.g., $(\partial f)^8$ and $(\partial^4 f)^2$ should be equally important. One analogy is with a massless scalar (e.g., dilaton) action in closed string theory: there is no known simple way to obtain a closed action involving only first derivatives of the dilaton; moreover, a summation of all terms with first derivatives only would contradict a low-energy expansion which is an expansion in powers of derivatives. Again, the case of the BI action is different: there, because of gauge invariance, the field strength F itself is playing the role of a fundamental field analogous

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Our aim here will be more modest: starting with the disc string partition function in the background (1.12) with $f = f(x_i)$, i.e., in the superstring case,

$$T = f(x_i) e^{\frac{x^0}{\sqrt{2}}} , \tag{1.16}$$

we shall compute directly the *first two* leading terms in Z, or in the corresponding effective lagrangian, in expansion in number of spatial derivatives

$$L = -\mathcal{Z} , \qquad \mathcal{Z} = V_0(T) + V_1(T)(\partial_i T)^2 + \cdots . \tag{1.17}$$

Here $\mathcal{Z}$ is the integral density in the partition function Z and dots stand for terms with more than two derivatives (assuming possibility of integration by parts). We shall use the methods and results of ref. [16] (confirming and generalizing some of them), which also emphasized (along with [14]) the importance of studying spatially-inhomogeneous tachyon backgrounds. We shall complement the expression (1.8) for the "potential" function V_0 found in [16] with the one for the "gradient" function V_1 in (1.17)

$$V_1 = \frac{1 - \ln(1 + \frac{1}{2}T^2)}{(1 + \frac{1}{2}T^2)^2} , \tag{1.18}$$

where we have set one ambiguous (field-redefinition dependent) coefficient to zero (see section 4).

This looks different than the coefficient appearing in (1.10) upon substituting (1.16) into (1.10)

$$L_{\rm KN} = -V_0(T)\left[1 + \frac{1}{2}(\partial_i T)^2 + \cdots\right] = -\frac{1}{1 + \frac{1}{2}T^2}\left[1 + \frac{1}{2}(\partial_i T)^2 + \cdots\right] , \tag{1.19}$$

but as in the case of (1.14) vs. (1.15) that could be attributed to a difference in scheme choices: the background that should correspond to (1.16) (viewed as a background for (1.10)) in the standard perturbative scheme used to compute Z should correspond to a particular case of (1.12) with *time-dependent* profile $f(x^0, x^i)$. Then to compare to (1.10) we would need to know also some time derivative dependent terms in (1.17) and their value on the corresponding T-background.

Given the near-on-shell nature of the background (1.16), it is natural to interpret $\mathcal{Z}$ as an effective potential energy produced on the D-brane by the tachyon profile function f. As we shall find below by explicitly computing the stress-energy tensor on the background (1.16), the energy of the system also changes sign at finite value of the tachyon field. The change of sign of the gradient function (1.18) from positive at $0 < |T| < T_*$ (where $T_* = \sqrt{2(e-1)} \approx 1.85$) to negative at $T_* < |T| < \infty$ which lowers the energy

to a massless scalar while ∂F is a counterpart of a scalar derivative, so F =const is always a solution. As a result, summing all orders in F while ignoring ∂F terms makes sense, while summing all orders in ∂F while ignoring all higher $\partial^n F$ terms would not.

suggests an instability of the system appearing at certain moment in time (i.e. at large enough value of T in (1.16)) — an instability towards creation of a spatial inhomogeneity $f \sim x$,[11] indicating an emergence of a codimension one D-brane.[12]

The rest of this paper is organized as follows.

We shall start in section 2 with a discussion of the bosonic string partition function in the corresponding analog $T = f(x_i)e^{x^0}$ of the background (1.16). We shall complement the result of [16] for the homogeneous case $f =$const with the expression for the first-derivative $O((\partial_i f)^2)$ term in Z.

In section 3 we shall turn to the superstring case. In section 3.1 we shall rederive the expression for V_0 in (1.8) [16] for the "homogeneous" (derivative-independent) part of $\mathcal{Z}$ in (1.17). We shall complement the discussion in [16] by (i) explaining why in the particular case of the background (1.16) one can indeed ignore (as was done in [16]) the contact T^2 term in the boundary part of the world-sheet action, and (ii) giving the general proof of the expression (1.8) to all orders in T (eq. (1.8) was checked in [16] only for the first few orders in expansion in powers of f). In section 3.2 we shall compute the gradient function V_1 (1.18) in (1.17).

In section 4 we will present the computation of the stress-energy tensor in the superstring background (1.16) to the second order in spatial derivative expansion. We will show that the condition of conservation of the stress-energy tensor is satisfied in the case of the marginal background (1.14). In section 5 we shall make some further comments on the implications of our result for $\mathcal{Z}$ (1.17) for the structure of the tachyon effective action.

Some technical details needed for the computation of the bosonic partition function in the e^{x^0} tachyon background will be given in appendix A. In appendix B we shall discuss a property of path ordered integral of a totally antisymmetric function used in section 3. Appendices C, D and E will provide some further details of the computation of the integrals appearing in the superstring case. Appendix F will contain a list of results used in the technically involved computation of the stress-energy tensor in section 4.

2. Bosonic string partition function

Before turning to the superstring case which is our main goal it is instructive to compute first the leading terms (1.17) in the partition function in the bosonic string case.

[11] This may be compared to the discussion in [2, 5] where a relevant tachyon perturbation $T = qx$ was interpreted as relating two conformal points $q = 0$ and $q = \infty$ through RG evolution, with $q \to \infty$ "freezing" the x-direction and thus representing a lower-dimensional brane. Here the evolution happens in *real* time $-\infty < x^0 < \infty$, and writing $T = (x-a)e^{\frac{1}{\sqrt{2}}x^0}$ one may interpret the $x^0 \to \infty$ region is the one where x is fixed at value a (transverse position of co-dimension one D-brane).

[12] The reason for this sign change in V_1 at *finite* value of T may be related to our neglect of higher-derivative contributions. The negative contribution of the gradient term to the energy is suppressed at large times since $V_1 \to 0$ at $T \to \infty$, so the energy at the end point of the evolution should be finite. We are grateful to A. Linde for a discussion of the issue of the sign change of V_1.

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2.1 General remarks

Our starting point will be the open bosonic string path integral on the disc with the boundary interaction term

$$I_{\rm bndy} = \int \frac{d\tau}{2\pi} T(x)\,, \tag{2.1}$$

where T is given by (1.12) with $\mu = 1$ and $f = f(x^i)$. For notational simplicity, we shall sometimes assume that the spatial profile function $f(x^i)$ depends only on coordinate $x^1 \equiv x$. Expanding the coordinates x^0, x^1 near constant (zero-mode) values $x^m \to x^m + X^m(\xi)$ and writing the interaction term in the string action as a Taylor expansion in powers of the fluctuation X^m we get

$$I_{\rm bndy}(x+X) = \int \frac{d\tau}{2\pi} e^{x^0+X^0}\left[f(x) + \partial f(x) X + \frac{1}{2}\partial^2 f(x) X^2 + \cdots\right]. \tag{2.2}$$

The general expression for the partition function is then

$$Z \sim \int dx^0 dx\, \mathcal{Z}(x^0, x)\,, \quad \mathcal{Z}(x^0,x) = \left\langle e^{-I_{\rm bndy}(x+X)}\right\rangle. \tag{2.3}$$

Here $\langle\cdots\rangle$ is the expectation value with the free string action on the disc. In the superstring case the string partition function Z on the disc is directly related to the massless mode (gauge vector) effective action, i.e. $S[A]$ is equal to (a renormalized value of) Z computed using $I_{\rm bndy} = \int d\tau A_m(x)\dot{x}^m$ [22, 21]; the same is expected to be true also in the tachyon case [5, 6, 23]. As for the bosonic case, here the relation between S and Z in the tachyon background case is less clear a priori; an expression for S (whose derivative should be proportional to the corresponding beta-function) suggested within boundary string field theory approach [24] is[13] $S = Z + \beta^T \frac{\delta Z}{\delta T}$, where $\beta^T = -T - \partial_m^2 T$ is the tachyon beta function. Here we shall consider only Z, i.e. will not study the corresponding bosonic effective action in detail.

To compute the partition function Z by expanding in powers of derivatives of f will require to know the correlators with arbitrary numbers of e^{X^0} insertions and fixed numbers of X-insertions like

$$\int d\tau_1 \cdots d\tau_n \left\langle e^{X^0(\tau_1)} \cdots e^{X^0(\tau_k)}\right\rangle \langle X(\tau_{k+1}) \cdots X(\tau_n)\rangle\,, \tag{2.4}$$

where respective correlators are evaluated with respective 2-d free actions, i.e. $\int d^2\xi (\partial X^0)^2$ and $\int d^2\xi(\partial X)^2$. We will therefore need to use and extend the methods of [16] who computed $\int d\tau_1 \cdots d\tau_n \langle e^{X^0(\tau_1)} \cdots e^{X^0(\tau_k)}\rangle$.

As a result, we expect to find (up to a total spatial derivative)

$$\mathcal{Z} = V_0(T) + U_1(T)(\partial_i T)^2 + U_2(T)\partial_i^2 T + \cdots = V_0(T) + V_1(T)(\partial_i T)^2 + \cdots\,. \tag{2.5}$$

[13]This is also a combination in which linear Möbius divergence in Z cancels out [5] (see also a discussion in [6]).

Here $T = T(x^0, x) = e^{x^0} f(x)$ and all derivatives are over the spatial coordinates x^i only, i.e. it is assumed that f and thus T are slowly varying in spatial direction. According to [16], the homogeneous ($\partial_i f = 0$) part of the bosonic partition function is

$$\mathcal{Z}_0 = V_0 = \frac{1}{1+T} . \tag{2.6}$$

Our aim will be to compute $V_1(T)$ in (2.5).

Let us first comment on some technical aspects of the computation of $\mathcal{Z}$. As usual, one expects to find 2-d divergences coming from contractions of the fluctuation fields X^m at the same point; these can be renormalized by a standard field redefinition of the tachyon field involving the beta-function (1.2), i.e. $\mathcal{Z}_{ren}$ $=\exp(\frac{1}{2}s_1\beta^T\frac{\partial}{\partial T})\mathcal{Z}$, where s_1 is a cut-off dependent (i.e. ambiguous) coefficient. There are no other divergences from coincident points — possible divergences coming from powers of propagators $\langle XX\rangle$ in (2.4) turn out to be suppressed by the contributions of the $\langle e^{X^0}\cdots e^{X^0}\rangle$ correlators. This may look surprising since in the bosonic string case one expects also power divergences corresponding to the Möbius infinities in the scattering amplitudes [22, 21]. The Möbius infinities are effectively hidden in the remaining integrals over the zero modes x^m: not performing integrals over x^m is equivalent to not imposing momentum conservation, and this effectively regularizes the Möbius infinities.[14]

Before turning to the general case, let us consider first the 2-point (order f^2) contribution to the two-derivative term in the partition function[15]

$$\mathcal{Z}_2 = (e^{x_0})^2[I_2(\partial f)^2 + I_2' f\partial^2 f] + \cdots , \tag{2.7}$$

i.e.

$$\mathcal{Z}_2 = -(\partial_i T)^2 + s_1 T\partial_i^2 T + \cdots . \tag{2.8}$$

We have used (2.6) and that

$$\begin{aligned} I_2 &= \frac{1}{2}\int \left\langle e^{X_0(\tau_1)}X(\tau_1)e^{X_0(\tau_2)}X(\tau_2)\right\rangle = -\frac{1}{\pi}\int_{-\pi}^{\pi} d\tau \sin^2\tau \ln(4\sin^2\tau) = -1 , \\ I_2' &= \frac{1}{2}\int \left\langle e^{X_0(\tau_1)}e^{X_0(\tau_2)}X^2(\tau_2)\right\rangle = -2\ln\epsilon = s_1 . \end{aligned} \tag{2.9}$$

Here $s_1 = \langle X^2(\tau)\rangle = G(\tau,\tau) = -2\ln\epsilon$ is a regularized propagator at coinciding points. The coefficient s_1 is ambiguous (field redefinition dependent). Adding $\mathcal{Z}_0$ (2.6) and integrating by parts gives, to quadratic order in T,[16]

$$\mathcal{Z} = 1 - T + T^2 - (1+s_1)(\partial_i T)^2 + \cdots . \tag{2.10}$$

[14]Consider, for example, a momentum non-conserving two-point function of the tachyons on the disc, i.e. $\int\langle V_{p_1}(z_1)V_{p_2}(z_2)\rangle \sim \int \frac{dz}{z^{1-\alpha' p^2}}$, where $p_1 + p_2 = p$. When $\alpha' p^2 \to 1$ the 2-point function diverges logarithmically which corresponds to a pole from the propagator of an intermediate state. For $p^2 \to 0$ the amplitude diverges linearly with a cut-off which represents the Möbius infinity.

[15]We have omitted the linear term $s_1 e^{x^0}\partial_i^2 f$ since being a total derivative in spatial directions it integrates to zero. It will be included in the general expression in the next subsection.

[16]If we define the effective action as $S = Z + \beta^T\frac{\partial Z}{\partial T}$ with the beta-function given in the present case by $\beta^T = -\partial_i^2 T$, we get, to quadratic order in T and $\partial_i T$: $L = 1 - T + T^2 - 2s_1(\partial_i T)^2 + \cdots$. This may be compared to the standard quadratic terms in the bosonic tachyon action (1.1) which for T given by (1.12) with $f = f(x^i)$ becomes simply $L = -\frac{1}{2}(\partial_i T)^2$.

Let us now generalize (2.10) to include all terms in expansion in powers of T but still keeping contributions with only two spatial derivatives of T.

2.2 Two-derivative term in bosonic partition function

To compute the complete two-derivative part of the partition function we shall use the method of orthogonal polynomials following [16] (see also [25]).[17] Some technical details of the computation are given in appendix A.

Let us concentrate on the finite $U_1(T)(\partial T)^2$ term in $\mathcal{Z}$ in (2.5). The $U_2(T)\partial^2 T$ term coming from contraction $\langle X^2 \rangle$ at one point has divergent (ambiguous) coefficient and can be eliminated by a field redefinition $f \to f + s_1 \partial^2 f$ in the potential term (2.6). The finite $(\partial f)^2$ term in $\mathcal{Z}$ is given by:

$$\mathcal{Z}_{2\ \mathrm{fin}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{2(n-2)!} I_n (e^{x^0})^n f^{n-2} (\partial f)^2 \,. \tag{2.11}$$

I_n is an integral of the Vandermonde determinant [16]

$$\Delta(\tau) = \prod_{i<j=1}^{n} (e^{i\tau_i} - e^{i\tau_j}) \,, \tag{2.12}$$

coming from $\langle e^{X^0(\tau_1)} \cdots e^{X^0(\tau_n)} \rangle$ with an insertion of logarithmic $\rangle XX \langle$ propagator:

$$I_n = -\int \prod_{i=1}^{n} \frac{d\tau_i}{2\pi} |\Delta(\tau)|^2 \ln\left[4 \sin^2\left(\frac{\tau_1 - \tau_2}{2}\right)\right] . \tag{2.13}$$

If we expand the logarithm in (2.13) in terms of cosines (as, e.g., in [20]) we can show that:

$$I_n = 2\int \prod_{i=1}^{n} \frac{d\tau_i}{2\pi} |\Delta(\tau)|^2 \sum_{m=1}^{\infty} \frac{1}{m} \cos m(\tau_1 - \tau_2) = -2(n-2)! \sum_{m=1}^{n-1} \frac{n-m}{m} \,. \tag{2.14}$$

The final result for the all-order form of the $(\partial f)^2$ term in (2.7) is given by substituting I_n into (2.11):

$$\mathcal{Z}_{2\ \mathrm{fin}} = -\sum_{n=2}^{\infty} (-1)^n (e^{x^0})^n f^{n-2} (\partial f)^2 \sum_{m=1}^{n-1} \frac{n-m}{m} \,. \tag{2.15}$$

The $n = 2$ term here agrees with (2.7). Observing that the resulting double sum can be written as a product of two series we finally get for the finite part of $\mathcal{Z}_2$ (with $T = e^{x^0} f(x^i)$):

$$\mathcal{Z}_{2\ \mathrm{fin}} = -\frac{\ln(1+T)}{T(1+T)^2} (\partial_i T)^2 \,. \tag{2.16}$$

Including also the derivative-independent term V_0 (2.6) and the divergent $\partial^2 f$ term (given by $\frac{1}{2} s_1 \frac{\partial V_0}{\partial T} \partial_i^2 T$) we finish with

$$\mathcal{Z} = \frac{1}{1+T}\left[1 - \frac{\ln(1+T)}{T(1+T)} (\partial_i T)^2 - \frac{s_1}{2(1+T)} \partial_i^2 T + \cdots \right] , \tag{2.17}$$

[17] We wish to thank F. Larsen and A. Naqvi for sharing with us their unpublished notes on the computations in [16].

or, after integration by parts,

$$\mathcal{Z} = \frac{1}{1+T}\left(1 - \left[\frac{\ln(1+T)}{T(1+T)} + \frac{s_1}{(1+T)^2}\right](\partial_i T)^2 + \cdots\right). \tag{2.18}$$

Expanded at small T this reduces to (2.10).

3. Superstring partition function

In the open superstring case the effective action should be directly equal to the (renormalized) disc partition function. Our aim here will be to compute the two leading terms (1.17) in spatial derivative expansion of the action by evaluating the superstring partition function Z in the background (1.12) (with $\mu = \mu_{\rm super} = \frac{1}{\sqrt{2}}$) or (1.16).

The derivative-independent part of Z was found in [16] to be equal to (1.8). There were some minor gaps in the derivation (cf. eqs.(61) and (67) and footnotes 6,7 in [16]) which we shall fill in below in section 3.1 (some technical details will be explained also in appendices B, C, D). In section 3.2 we shall compute the second-derivative term in Z, obtaining the analog of the bosonic expression in (2.18) (see also appendix E).

3.1 Derivative-independent ("potential") term

The starting point is the world-sheet supersymmetric expression for the tachyon coupling in the open NS string [26, 2, 5] (see also [6, 27])

$$I_{\rm bndy} = \int \frac{d\tau}{2\pi} d\theta [\hat\zeta D\hat\zeta + \hat\zeta T(\hat x)] = \int \frac{d\tau}{2\pi}[\zeta\dot\zeta + \zeta\psi^m\partial_m T(x) + hT(x) + h^2]\,, \tag{3.1}$$

where $\hat\zeta$ and $\hat x^m$ are 1-d superfields with components ζ, h and x^m, ψ^m. Integrating out h we are left with

$$I_{\rm bndy} = -\frac{1}{4}\int \frac{d\tau}{2\pi}(T^2 - \psi^m\partial_m T\partial_\tau^{-1}\psi^n\partial_n T)\,. \tag{3.2}$$

Computing the path integral in the supersymmetric form starting with (3.1) as in, e.g., [21] we will need to use

$$\begin{aligned}\left\langle \hat\zeta(\tau_i,\theta_i)\hat\zeta(\tau_j,\theta_j)\right\rangle &= \hat\Theta(i,j) \equiv \Theta(\tau_i - \tau_j + \theta_i\theta_j) = \Theta(i,j) + \theta_i\theta_j\delta(i,j)\,,\\ \Theta(i,j) &\equiv \Theta(\tau_i - \tau_j)\,, \qquad \delta(i,j) \equiv \delta(\tau_i - \tau_j)\,,\end{aligned} \tag{3.3}$$

where $\Theta(\tau)$ is a step function.

Starting with (3.1) with the tachyon coupling $T = e^{\frac{x^0}{\sqrt{2}}} f(x)$ we find for the ∂f derivative-independent term in the superstring partition function

$$\begin{aligned}\mathcal{Z}_0 = &\sum_{n=0}^{\infty}(-1)^n\left(e^{\frac{x^0}{\sqrt{2}}}\right)^{2n} f^{2n}\int\prod_{i=1}^{2n}\frac{d\tau_i}{2\pi}d\theta_i\hat\Theta(1,2)\ldots\hat\Theta(2n-1,2n)\times\\ &\times\prod_{i<j}^{2n}|e^{i\tau_i} - e^{i\tau_j} - ie^{i(\tau_i+\tau_j)}\theta_i\theta_j|\,.\end{aligned} \tag{3.4}$$

One can show that here the contact $\delta(i,j)$ part of the supersymmetric theta-functions drops out since it gives zero whenever it is picked up by the $d\theta_i$ integrations. Equivalently, the "contact" T^2-term in the component form of the boundary interaction term (3.2) can be omitted since it does not contribute to the final result. This explains why the result of [16] where the T^2-term was not included from the very beginning is indeed the same as obtained using the manifestly world-sheet supersymmetric boundary interaction (3.1).

Integrating over θ_i we finish with the following expression for the coefficient of the n-th term in the sum (3.4):

$$J_n = \int [d\tau]_{2n} \sum_{\mathcal{P}} (-1)^{\mathcal{P}(i_1,\ldots,i_{2n})} \epsilon(i_1,i_2)\cdots\epsilon(i_{2n-1},i_{2n}) \frac{\prod_{i<j}^{2n} G(i,j)}{G(i_1,i_2)\cdots G(i_{2n-1},i_{2n})} \quad (3.5)$$

$$[d\tau]_{2n} \equiv \prod_{i=1}^{2n} \frac{d\tau_i}{2\pi} \Theta(1,2)\cdots\Theta(2n-1,2n)\,. \quad (3.6)$$

We used that $|e^{i\tau_i} - e^{i\tau_j} - ie^{i(\tau_i+\tau_j)}\theta_i\theta_j| = |e^{i\tau_i} - e^{i\tau_j}| + \epsilon(\tau_i - \tau_j)\theta_i\theta_j$, where $\epsilon(\tau)$ is the sign function.

Equivalently,

$$J_n = \int [d\tau]_{2n} \prod_{i<j}^{2n} \epsilon(i,j) W(1,\ldots,2n)\,, \quad (3.7)$$

$$W(1,\ldots,2n) \equiv \sum_{\mathcal{P}} (-1)^{\mathcal{P}(i_1,\ldots,i_{2n})} \frac{\prod_{i<j}^{2n} D(i,j)}{D(i_1,i_2)\cdots D(i_{2n-1},i_{2n})}\,. \quad (3.8)$$

Here we used the following notation:

$$\epsilon(i,j) = \epsilon(\tau_i - \tau_j) = \Theta(i,j) - \Theta(j,i)\,, \qquad G(i,j) = |e^{i\tau_i} - e^{i\tau_j}| = \epsilon(i,j)D(i,j)\,, \quad (3.9)$$

$$D(i,j) = i[e^{i(t_i-t_j)/2} - e^{-i(t_i-t_j)/2}] = 2\sin(\frac{t_i - t_j}{2})\,. \quad (3.10)$$

$\mathcal{P}(i_1,\ldots,i_{2n})$ means all $(2n-1)!!$ permutations of ordered pairs of the $2n$ indices.[18]

One can check that $W(1,2,\ldots,2n)$ in (3.7) is symmetric under all interchanges of the arguments. The factor $\prod_{i<j}\epsilon(i,j)$ is of course antisymmetric. In appendix B we show that for a totally antisymmetric function $A(1,\ldots,2n) \equiv A(\tau_1,\ldots,\tau_{2n})$ there is the following relation

$$\int [d\tau]_{2n} A(1,\ldots,2n) = \frac{1}{(2n)!} \int \prod_{i=1}^{2n} \frac{d\tau_i}{2\pi} \prod_{i<j} \epsilon(i,j) A(1,\ldots,2n)\,. \quad (3.11)$$

Then combining (3.5) and (3.11) we can get rid of the $\epsilon(i,j)$ factors and find

$$\mathcal{Z}_0 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(e^{\frac{x^0}{\sqrt{2}}}\right)^{2n} f^{2n} \int \prod_{i=1}^{2n} \frac{d\tau_i}{2\pi} W(1,2,\ldots,2n)\,. \quad (3.12)$$

[18] For example, in the case of $n = 3$ we have schematically (with sign factors included): $\{(1,2);(3,4);(5,6)\} \to (-)\{(1,3);(2,4);(5,6)\} \to (+)\{(1,4);(2,3);(5,6)\}$, etc.

Finally, using eq. (D.3) of appendix D, i.e. $\int\prod_{i=1}^{2n}\frac{d\tau_i}{2\pi}W(1,\dots,2n)=(2n-1)!!n!$, we can show that the total result for $\mathcal{Z}_0$ is indeed the one (1.8) of ref. [16], i.e.

$$\mathcal{Z}_0 = V_0(T) = \frac{1}{1+\frac{1}{2}T^2}\,, \qquad T = e^{\frac{x^0}{\sqrt{2}}}f(x)\,. \tag{3.13}$$

3.2 Two-derivative ("gradient") term

Next, let us compute the two-derivative term in $\mathcal{Z}$ and thus in the effective action. Expanding the boundary interaction term (3.1) near constant values of the coordinates, $\hat{x}^m = x^m + \hat{X}^m$, where $\hat{X}^m(\tau,\theta)$ is a fluctuation superfield, one has

$$\int\frac{d\tau}{2\pi}d\theta\hat{\zeta}T(\hat{x}) = \int\frac{d\tau}{2\pi}d\theta\hat{\zeta}e^{\frac{x^0+\hat{X}^0}{\sqrt{2}}}\left[f(x)+\hat{X}\partial f(x)+\frac{1}{2}\hat{X}^2\partial^2 f+\cdots\right]. \tag{3.14}$$

As in the bosonic case, the contraction of the two $\hat{X}$-fields at the same point produces a logarithmic divergence that can be renormalized away by a redefinition of the tachyon coupling, i.e. the $\partial^2 f$ term enters $\mathcal{Z}$ with an ambiguous coefficient:

$$\mathcal{Z}_{2\ amb.} = \frac{1}{2}s_1\frac{\partial\mathcal{Z}_0}{\partial T}\partial_i^2 T\,. \tag{3.15}$$

After using (3.13) and integrating by parts we get (cf. (2.17),(2.18))

$$\mathcal{Z}_{2\ amb.} = \frac{1}{2}s_1\frac{1-\frac{3}{2}T^2}{(1+\frac{1}{2}T^2)^3}(\partial_i T)^2\,. \tag{3.16}$$

The finite (unambiguous) part of $\mathcal{Z}_2$ is given by

$$\mathcal{Z}_{2\ \mathrm{fin}} = -\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}J_n\left(e^{\frac{x^0}{\sqrt{2}}}\right)^{2n} f^{2n-2}(\partial_i f)^2\,,$$

$$J_n = \int\prod_{i=1}^{2n}\frac{d\tau_i}{2\pi}d\theta_i\ \hat{\Theta}(1,2)\dots\hat{\Theta}(2n-1,2n)\times \tag{3.17}$$

$$\times\prod_{i<j}^{2n}|D(i,j)|\sum_{k<l}^{2n}\left(\ln|D(k,l)|+\frac{\epsilon(k,l)}{|D(k,l)|}\theta_k\theta_l\right). \tag{3.18}$$

There are two types of terms in the integrand of the above expression. The first one involving logarithms can be shown to be antisymmetric, and then using the relation in appendix B we can replace the path ordered integral by an ordinary integral. The second one requires a little more work but after doing the $d\theta_i$ integrations it turns out to be proportional to the integrand of $\mathcal{Z}_0$ in (3.5), and at the end one gets again an ordinary integral. After some manipulations and using symmetry of the integrand we find

$$\mathcal{Z}_{2\ \mathrm{fin}} = -\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)!}C_n\left(e^{\frac{x^0}{\sqrt{2}}}\right)^{2n} f^{2n-2}(\partial_i f)^2\,, \tag{3.19}$$

$$C_n = \frac{1}{2n}J_n = \int\prod_{i=1}^{2n}\frac{d\tau_i}{2\pi}\ W(1,2,\dots,2n)[1+(2n-1)\ln|D(1,2)|]\,. \tag{3.20}$$

Expanding the logarithm in (3.20) in a power series of cosines (as in the bosonic case in (2.14)) gives

$$C_n = \int \prod_{i=1}^{2n} \frac{d\tau_i}{2\pi}\, W(1,2,\ldots,2n) \left[1 - (2n-1)\sum_{m=1}^{\infty} \frac{1}{m}\cos m(\tau_1 - \tau_2)\right]. \tag{3.21}$$

The first term in the square brackets gives the same integral as in the previous subsection (cf. (3.12)). As explained in appendix (E), the second term is also evaluated following the same reasoning as in the zero-derivative case: one is to expand the cosines in polynomials of exponents and use the relations of appendix D. We find

$$C_n = (2n-1)!!(n-1)!(\sum_{m=1}^{n-1} \frac{n-m}{m} + n)\,. \tag{3.22}$$

One can check that the direct calculation of (3.21) for $n = 1$ and $n = 2$ gives the values $C_1 = 1$ and $C_2 = 9$ respectively, in agreement with (3.22). Finally, after similar manipulations as in the bosonic case we get for the finite part of $\mathcal{Z}_2$:

$$\mathcal{Z}_{2\ \mathrm{fin}} = \frac{1}{(1+\frac{1}{2}T^2)^2}\left[1 - \ln\left(1 + \frac{1}{2}T^2\right)\right](\partial_i T)^2\,. \tag{3.23}$$

For example, the first two terms of the small T expansion of (3.23) are: $(\partial_i T)^2 - \frac{3}{2}T^2(\partial_i T)^2$, in agreement with (3.19) with $C_1 = 1$ and $C_2 = 9$.

Adding the potential (3.13) and the ambiguous (3.16) terms to (3.23) we end up with the central result of this paper — the expression for the superstring partition function to the second order in spatial derivatives of the tachyon:

$$\mathcal{Z} = \frac{1}{1+\frac{1}{2}T^2}\left(1 + \frac{1}{1+\frac{1}{2}T^2}\left[1 - \ln(1+\frac{1}{2}T^2) + \frac{1}{2}s_1\frac{1-\frac{3}{2}T^2}{1+\frac{1}{2}T^2}\right](\partial_i T)^2 + \cdots\right). \tag{3.24}$$

This may be compared to the bosonic string result (2.18).

3.3 Including x^0 dependence in f

As was already mentioned in the Introduction, in the case of $f = f(x^i, x^0)$ which is close to a marginal perturbation (which should satisfy (1.13)) the expansion in derivatives should be organized so that to take into account that $\partial_0 f \sim \partial_i^2 f \sim (\partial_i f)^2$. Indeed, expanding in time derivatives of f we get

$$\mathcal{Z}_{fin} = V_0(T) + V_1(T)\left(e^{\frac{x^0}{\sqrt{2}}}\right)^2(\partial_i f)^2 + K_1(T)e^{\frac{x^0}{\sqrt{2}}}\partial_0 f + O((\partial_i f)^4, (\partial_i^2 f)^2, (\partial_0 f)^2, \partial_0 f(\partial_i f)^2)\,. \tag{3.25}$$

Note that expansion in time derivatives of T does not make sense: using that $e^{\frac{x^0}{\sqrt{2}}}\partial_0 f = \partial_0 T - \frac{1}{\sqrt{2}}T$ one concludes that if one expands in $\partial_0 T$, then coefficients of lower-derivative terms receive contributions from all higher-derivative $\partial_0^n f$, etc., terms.[19]

[19]In fact, expansion in powers of time derivatives of f is also not well defined: because of time-dependent coefficients in (3.25), terms with different powers of ∂_0 may mix in the resulting equations of motion.

Computing K_1 in (3.25) one gets:

$$K_1 = -\sqrt{2} T V_1(T) \,. \tag{3.26}$$

Similarly, one can compute with some effort the coefficients of the next-order terms $(\partial_i f)^4$, $(\partial_i^2 f)^2$, $(\partial_0 f)^2$, $\partial_0 f (\partial_i f)^2$. They happen to contain second powers of $\ln(1 + \frac{1}{2}T^2)$, which may be prompting a possibility of some resummation of the derivative expansion.

4. Stress-energy tensor

In this section we will evaluate the target-space stress-energy tensor (SET) in the superstring background (1.16) to first order in spatial derivative of f. For this we will need to compute the expectation value of the graviton vertex operator in the background (1.16).

As was explained in [16], the SET can be found from the following expression

$$T^{mn} = K[\mathcal{Z}(x^k)\eta^{mn} + A^{mn}(x^k)] \,, \tag{4.1}$$

where K is an overall normalization constant, $\mathcal{Z}$ is the partition function density and

$$A^{mn} \equiv \left\langle : V^{mn}(0,0) : e^{-I_{\text{bndy}}} \right\rangle = W^{mn} + \eta^{mn}\mathcal{Z}(x^k) \,. \tag{4.2}$$

Here the graviton vertex operator is fixed on the center of the disc[20] and has the form:

$$V^{mn} = 2 \int d\theta d\bar{\theta} D\hat{X}^m(0) \bar{D}\hat{X}^n(0) \,. \tag{4.3}$$

We shall define as in [16] the following modified normal ordering:

$$\mathring{\circ} V^{mn} \mathring{\circ} =: V^{mn} : -\eta^{mn} \,. \tag{4.4}$$

Then

$$W^{mn} \equiv \frac{1}{2} \left\langle \mathring{\circ} V^{mn}(0,0) \mathring{\circ}\ e^{-I_{\text{bndy}}} \right\rangle , \tag{4.5}$$

and so (4.1) can be written as

$$T^{mn} = 2K[\eta^{mn}\mathcal{Z}(x^k) + W^{mn}(x^k)] \,. \tag{4.6}$$

We will begin with the computation of the W^{0i} component which is the easiest:

$$\begin{aligned} W^{0i} = \sum_{n=0}^{\infty} (-1)^n \left(e^{\frac{x^0}{\sqrt{2}}} \right)^{2n} \int d\mu_n \Bigg\langle \mathring{\circ} \int d\theta d\bar{\theta} D\hat{X}^0(0) \bar{D}\hat{X}^i(0) \mathring{\circ} \times \\ \times \prod_{l=1}^{2n} e^{\frac{\hat{X}^0(z_l)}{\sqrt{2}}} \left[f^{2n} + f^{2n-1}\partial_i f \sum_{m=1}^{2n} \hat{X}^i(z_m) + \cdots \right] \Bigg\rangle , \end{aligned} \tag{4.7}$$

where

$$d\mu_n = \int \prod_{i=1}^{2n} \frac{d\tau_i}{2\pi} d\theta_i \hat{\Theta}(1,2) \cdots \hat{\Theta}(2n-1,2n) \,. \tag{4.8}$$

[20] Fixing the position of the graviton vertex can be always done in the conformal background using Möbius symmetry. Away from conformal points this represents a particular "off-shell" definition of the stress tensor.

We use the results of [16] for the SET computation in the f=const background.[21] Only the second term in the parentheses contributes and we are lead to the result

$$W^{0i} = \sum_{n=0}^{\infty} (-1)^{n+1} (e^{\frac{x^0}{\sqrt{2}}})^{2n} \frac{\sqrt{2}}{2^n} f^{2n-1} \partial_i f = \frac{1}{\sqrt{2}} \frac{T\partial_i T}{1+\frac{T^2}{2}} . \tag{4.9}$$

Inserting this into (4.6) we get

$$T^{0i} = 2KW^{0i} = \sqrt{2}K \frac{T\partial_i T}{1+\frac{T^2}{2}} . \tag{4.10}$$

Next, let us compute W^{ij}:

$$\begin{aligned} W^{ij} = & \sum_{n=0}^{\infty} (-1)^n \left(e^{\frac{x^0}{\sqrt{2}}}\right)^{2n} \int d\mu_n \times \\ & \times \Big\langle \circ\!\!\circ \int d\theta d\bar{\theta} D\hat{X}^i(0) \bar{D}\hat{X}^j(0) \circ\!\!\circ \\ & \quad \times \prod_{l=1}^{2n} e^{\frac{\hat{X}^0(z_l)}{\sqrt{2}}} \Bigg[f^{2n} + f^{2n-1}\partial_a f \sum_{j=1}^{2n} \hat{X}^a(z_j) + f^{2n-2}\partial_a f \partial_b f \sum_{l<m} \hat{X}^a(z_l)\hat{X}^b(z_m) + \\ & \qquad + \frac{1}{2} f^{2n-1} \partial_a\partial_b f \sum_{m=1}^{2n} \hat{X}^a(z_m)\hat{X}^b(z_m) + \cdots \Bigg] \Big\rangle . \end{aligned} \tag{4.11}$$

The first two terms in the square brackets do not contribute. The remaining integrals are again of the same type as in [16] and lead to the expression

$$W^{ij} = \sum_{n=1}^{\infty} (-1)^n \left(e^{\frac{x^0}{\sqrt{2}}}\right)^{2n} \frac{f^{2n-2}}{2^n} [\eta^{ij}(2-2n)(\partial_k f)(\partial_k f) + 2nf\partial_i\partial_j f] . \tag{4.12}$$

Plugging this into (4.6) and using the partition function expression (3.24) from the previous section we get

$$\begin{aligned} T^{ij} = 2K \Bigg[& \eta^{ij} \frac{1}{1+\frac{1}{2}T^2} + \eta^{ij}(\partial_k T)^2 \left(\frac{1-\ln(1+\frac{1}{2}T^2)}{(1+\frac{1}{2}T^2)^2} + \frac{1}{2}s_1 \frac{1-\frac{3}{2}T^2}{(1+\frac{1}{2}T^2)^3} \right) \\ & + (\partial_i T)(\partial_j T) \frac{-\frac{1}{2}T^2}{(1+\frac{1}{2}T^2)^2} + T(\partial_i\partial_j T) \frac{-1}{(1+\frac{1}{2}T^2)^2} + \cdots \Bigg] . \end{aligned} \tag{4.13}$$

where $T = e^{\frac{x^0}{\sqrt{2}}} f(x_i)$.

It is straightforward to check that the above two components T^{0i} (4.10) and T^{ij} (4.13) satisfy the SET conservation law

$$\partial_0 T^{0i} + \partial_j T^{ji} = 0 + O((\partial T)^3) . \tag{4.14}$$

Here only the first term in T^{ij} in (4.13) is taken into account ($(\partial_i T)^2$ terms lead to $O((\partial T)^3)$ corrections which we ignore).

[21] We have also verified equations (67) in [16] for all n using our method.

The computation of W^{00} turns out to be long and complicated so we will only outline some basic steps and leave details for appendix F. We start with

$$W^{00} = \sum_{n=0}^{\infty}(-1)^n(e^{\frac{x^0}{\sqrt{2}}})^{2n}\int d\mu_n \times$$
$$\times\left\langle {}^{\circ}_{\circ}\int d\theta d\bar{\theta} D\hat{X}^0(0)\bar{D}\hat{X}^0(0){}^{\circ}_{\circ}\right.$$
$$\times\prod_{l=1}^{2n} e^{\frac{\hat{X}^0(z_l)}{\sqrt{2}}}\left[f^{2n}+f^{2n-1}\partial_i f\sum_{m=1}^{2n}\hat{X}^i(z_m)+f^{2n-2}\partial_k f\partial_j f\sum_{l<m}\hat{X}^k(z_l)\hat{X}^j(z_m)+\right.$$
$$\left.\left.+\frac{1}{2}f^{2n-1}(\partial_i\partial_j f)\sum_{m=1}^{2n}\hat{X}^i(z_m)\hat{X}^j(z_m)+\cdots\right]\right\rangle . \tag{4.15}$$

The first term in the square brackets is easy to evaluate since it is the same integral as in [16]. Its contribution is $\mathcal{Z}_0(T)-1$. The second term does not contribute. The last one leads to a logarithmically divergent term which after a renormalization gives an ambiguous contribution

$$W^{00}_{amb} = s_1\sum_{n=1}^{\infty}(-1)^n\left(e^{\frac{x^0}{\sqrt{2}}}\right)^{2n}\frac{f^{2n-1}}{2^n}n\partial^2 f = \frac{1}{2}s_1\partial^2 T\frac{\partial\mathcal{Z}_0}{\partial T}\,. \tag{4.16}$$

In the third term we first do the integration over the fermionic coordinates θ and use (as in the previous sections) the symmetry of the integrand to get rid of the Θ functions. This leads to the following integral

$$I_{2n} = -\frac{1}{(2n)!}\int\prod_{l=1}^{2n}\frac{d\tau_l}{2\pi}W(1,2,\ldots 2n)\left[\sum_{i<j}\ln|e^{i\tau_i}-e^{i\tau_j}|^2+2n\right]\left[n+\sum_{i<j}\cos(\tau_i-\tau_j)\right]. \tag{4.17}$$

The contribution of the first term in the second brackets is proportional to (3.22) and leads to

$$N_{2n} = -\frac{n}{2^{n-1}}\left[\sum_{m=0}^{n-2}\frac{n-m-1}{m+1}\Theta(n-2)+n\Theta(n-1)\right]. \tag{4.18}$$

The product of the second term in the second brackets in (4.17) with the second term in the first brackets leads to an integral similar to the one in [16] and gives

$$L_{2n} = -\frac{1}{2^{n-1}}n(1-n)\Theta(n-1)\,. \tag{4.19}$$

The remaining complicated contribution is evaluated in appendix F. It reads

$$M_{2n} = \frac{1}{2^{n-1}}\left[(n-1)\sum_{m=0}^{n-2}\frac{n-m-1}{m+1}-\frac{1}{2n}\Theta(n-1)+\Theta(n-2)+\delta_{n,1}\right]. \tag{4.20}$$

Combining (4.18),(4.19),(4.20) gives $I_{2n}=N_{2n}+L_{2n}+M_{2n}$ in (4.17). Plugging it into (4.15) we get

$$W^{00} = (\partial_i f)^2\sum_{n=1}^{\infty}(-1)^n(e^{\frac{x^0}{\sqrt{2}}})^{2n}(\frac{-f^2}{2})^{n-1}[\sum_{m=0}^{n-2}\frac{n-m-1}{m+1}\Theta(n-2)+\frac{1}{2n}+(n-1)\Theta(n-2)]\,. \tag{4.21}$$

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Adding non-derivative part of (4.15) we get finally

$$W^{00} = \mathcal{Z}_0 - 1 + (\partial_i T)^2 [(\frac{1}{T^2} - \frac{T^2}{2(1+\frac{1}{2}T^2)^2}) \ln(1+\frac{1}{2}T^2) - \frac{T^2}{2(1+\frac{1}{2}T^2)^2}] . \tag{4.22}$$

The 00 component of the SET thus becomes

$$T^{00} = 2K \Big[-1 + \Big[\frac{1}{T^2} \ln\Big(1+\frac{1}{2}T^2\Big) - \frac{1}{1+\frac{1}{2}T^2} \Big] (\partial_i T)^2 + O((\partial T)^3) \Big] . \tag{4.23}$$

The value of the normalization constant K can be determined by considering the limit of $f =$const: then K can be interpreted as the tension of the original D-brane $K = \frac{1}{2}\mathcal{T}_p$ [16].

The conservation of SET requires

$$\partial_0 T^{00} + \partial_i T^{i0} = 0 . \tag{4.24}$$

Ignoring higher-derivative terms, that leads in the present case to the equation:

$$\frac{T}{(1+\frac{1}{2}T^2)} \partial_i^2 T = 0 + O((\partial T)^3) . \tag{4.25}$$

For $T = f(x_i) e^{\frac{x^0}{\sqrt{2}}}$ this equation is equivalent to the leading-order marginality condition $\partial_i^2 f = 0$. This is the expected conclusion since the conservation of the SET should be automatic for conformally-invariant backgrounds. The condition $\partial_i^2 f = 0$ is solved by the linear background

$$f(x) = f_0 + q_i x^i , \tag{4.26}$$

or T in (1.14).

Let us comment on the physical interpretation of the rolling tachyon background with a linear spatial profile (4.26),(1.14). By a global rotation we can always set $q_i x^i = ax$, $x = x_1$, i.e. it is sufficient to consider a "one-dimensional" inhomogeneity, $f = qx$ (we can absorb f_0 into x). In ref. [2, 5] it was shown that the spatial (time-independent) linear tachyon perturbation $T(x) = qx$ results in an RG flow for the coefficient q from zero to infinity, which effectively changes the boundary condition in the x-direction from the Neumann to the Dirichlet one. It is natural to expect that the target-space time x^0 evolution from $-\infty$ to $+\infty$ in the present case of

$$T = q(x^0)x , \quad q(x^0) = q e^{\frac{x^0}{\sqrt{2}}} \tag{4.27}$$

simulates this situation in the "on-shell" case where the world-sheet theory remains conformal throughout the time evolution. In ref. [28] the change of sign of the stress-energy tensor for a critical value of the tachyon field T_{cr} was associated with an emergence of a codimension one D-brane from the rolling tachyon decay of a non-BPS D-brane. A similar sign change happens in our case where the energy density T^{00} in (4.23) passes through zero at some value of time (for fixed q and x). A particular location of the space-time point where the sign change occurs is an artifact of the derivative expansion. One may expect that this location will move to $T = \infty$ region once higher-derivative corrections to the stress-energy tensor are included.

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5. Concluding remarks

The superstring partition function (3.24) found in section 4 may be interpreted as giving the value of the corresponding effective lagrangian evaluated on the inhomogeneous rolling tachyon background $T = f(x_i)e^{\frac{x^0}{\sqrt{2}}}$ to the second order in spatial derivatives of the tachyon

$$L = -\mathcal{Z} = -\frac{1}{1+\frac{1}{2}T^2}\left(1+\frac{1}{1+\frac{1}{2}T^2}\left[1-\ln\left(1+\frac{1}{2}T^2\right)+\frac{1}{2}s_1\frac{1-\frac{3}{2}T^2}{1+\frac{1}{2}T^2}\right](\partial_i T)^2+\cdots\right), \quad (5.1)$$

where s_1 can be changed by a field redefinition. As already mentioned in the Introduction, this may be related to the TDBI action (1.10) evaluated on the same tachyon profile (1.19) by a complicated field redefinition involving time derivatives of the tachyon. This issue requires further study.

Expanding (5.1) at small T we get

$$L \approx -1 + \frac{1}{2}T^2 - \frac{1}{4}T^4 - \left(1+\frac{1}{2}s_1\right)(\partial_i T)^2 + \cdots . \quad (5.2)$$

For large T

$$L \approx -\frac{2}{T^2}\left[1+\frac{2}{T^2}(c_1 - 2\ln T)(\partial_i T)^2 + \cdots\right], \qquad c_1 = 1 + \ln 2 - \frac{3}{2}s_1 . \quad (5.3)$$

The direct variation of (5.1) over spatial tachyon profile does not lead to the expected leading-order marginality (i.e. beta-function, cf. (1.13)) equation, $\partial_i^2 f = 0$; in particular, (5.1) does not have the linear background (4.26) as its exact solution. At the same time, the condition of conservation of stress-energy tensor (4.25) did lead us to the correct on-shell condition in the two-derivative approximation.

This may look puzzling, but has a simple explanation. To be able to derive the correct equations of motion one needs to compute first the partition function for the general profile function $f(x_0, x_i)$ depending also on the time direction. The reason is that since our tachyon background (1.16) depends on time, the time derivatives which are acting on $f(x_0, x_i)$ in the full action may become acting on T-dependent factors in the equation of motion, producing new terms compared to the case where one starts with the action depending only on $f(x_i)$. For example, if the action contains the term $T^n \partial_0^2 f(x_0, x_i)$ where $T = f(x_0, x_i)e^{x^0/\sqrt{2}}$, then in the equation of motion we may get a term $\partial_0^2 T^n$ which may give a non-zero contribution even after we replace $f(x_0, x_i)$ by $f(x_i)$.

A somewhat related comment applies to the expression for the stress-energy tensor found in section 4. One may wonder if T^{mn} (or at least its spatial components) can be obtained from a covariantization of (5.1) and a variation over the background metric (as, e.g., in [28, 18]). This is not the case: considering a generalization of (5.1) to a constant spatial metric and taking derivative over it we get an expression similar but not exactly equal to (4.13). One possible reason for this discrepancy is that tachyon action on a curved background may contain terms depending on derivatives of the metric, e.g.,

$g(T)R+h(T)D^i\partial_i T$. Then taking a variation of the action over the metric and then setting the metric to be flat one may produce additional tachyon derivative terms in T^{ij}.[22]

As we discussed in the Introduction, there is no direct Lorentz-covariant extension of Z computed in *derivative expansion* at the vicinity of (1.16). Still, it may be of interest to study a "model" covariant action obtained from the partition function (5.1) by simply replacing $\partial_i T \to \partial_m T$.

Then (5.2) with $s_1 = 0$ will agree with the standard quadratic tachyonic action (1.1) (up to an overall normalization $\frac{1}{2}$), and so one would reproduce the correct tachyon equation of motion to linear order in T.[23] We then get a direct covariantization of (1.17)

$$L = -V_0(T) - V_1(T)(\partial_m T)^2 \,, \tag{5.4}$$

where for $s_1 = 0$ in (5.1)

$$V_0 = \frac{1}{1+\frac{1}{2}T^2} \,, \qquad V_1(T) = \frac{1-\ln(1+\frac{1}{2}T^2)}{(1+\frac{1}{2}T^2)^2} \,. \tag{5.5}$$

As was already discussed in the Introduction, the change of sign of the kinetic function V_1 in (5.4) at $T = \pm|T_*| \approx 1.85$ suggests development of a spatial inhomogeneity. We arrived at a qualitatively similar conclusion in the analysis of the stress-energy tensor in the previous section. In both cases higher derivative corrections are expected to push the location of the transition point to $T \sim \infty$, i.e. $x^0 \to \infty$ region.

Redefining T to get the standard kinetic term in the region $0 < |T| < |T_*|$, i.e. $L = -(\partial_m T')^2 - V(T')$, one finds the potential $V(T')$ that changes from a maximal value 1 at the tachyonic vacuum at $T = T' = 0$ to its minimal value 0.368 at $T'(T_*) = 1.04$. Alternatively, in the region $|T_*| \ll |T| < \infty$ one may redefine T to try connect the resulting action to the one in (1.3). Indeed, in the large T limit the lagrangian (5.1) or (5.3) becomes similar (but not equivalent) to (1.3) (after a field redefinition $T^2 \to e^{\frac{1}{4}T^2}$).

Finally, let us mention again that it would be interesting to compute the exact expression for the superstring partition function on some special spatially-inhomogeneous backgrounds which are exactly marginal. In particular, for the linear profile one $T = (f_0 + q_i x^i) e^{\frac{x^0}{\sqrt{2}}}$ discussed above it is easy to show that the corresponding partition function contains no 2-d UV divergences to all orders in expansion in q_i, suggesting that this is indeed an exactly marginal background.[24] Computing the exact dependence of $\mathcal{Z}$ on q_i is equivalent to fixing its dependence on all powers of the gradient $\partial_i f$ in $T = f(x_i) e^{\frac{x^0}{\sqrt{2}}}$. Unfortunately, in contrast to the case of the simplest "off-shell" tachyon background $T = q_i x^i$ in [5] which leads to a gaussian world-sheet theory, it is not clear at the moment how to compute explicitly the coefficient of generic q^n term in expansion of $\mathcal{Z}$ in powers of q_i.

[22] It may be of interest to verify this expectation by computing the three-point scattering amplitude for a graviton and two tachyons to extract the coupling of gravity to order T^2 and thus to compare it with (5.1).

[23] At the same time, if one uses the replacement $(\partial_i T)^2 \to (\partial_m T)^2 + \frac{1}{2}T^2$ which is the identity for the near-mass-shell tachyon background (1.16) we have considered, then to match (1.1) one would need to set $s_1 = -1$. In this case the potential term and the coefficient of the $(\partial_m T)^2$ term may get contributions also from higher-order terms in spatial derivative expansion which we ignored in (3.24).

[24] The logarithmic singularities from correlators of x_i at coinciding points are always suppressed by zeroes from correlators of $e^{\frac{x^0}{\sqrt{2}}}$; in particular, the OPE of two of such operators is "soft".

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A. Method of orthogonal polynomials for computing integrals in the bosonic case

In this appendix we discuss some details of the computation of the integral which appears in eq. (19) of [16]:

$$\int \prod_{i=1}^{n} \frac{d\tau_i}{2\pi} |\Delta(\tau)|^2 = n!\,, \tag{A.1}$$

where the Vandermonde determinant $\Delta(\tau)$ is:

$$\Delta(\tau) = \prod_{i<j}^{n} (e^{i\tau_i} - e^{i\tau_j}) = \sum_{\Pi\{i_k\}} \prod_{l=0}^{n-1} (-1)^{\Pi} e^{i\Pi(l)\tau_l} = \sum_{\Pi\{i_k\}} \prod_{k=1}^{n} (-1)^{\Pi} P^{k-1}(\lambda_{i_k})\,. \tag{A.2}$$

Π stands for all permutations of the $\{i_k\}$ indices and the polynomials $P^m(\lambda_k) \equiv P^m(k)$ are defined by

$$P^m(k) \equiv P^m(\lambda_k) = \lambda_k^m\,, \quad \lambda_k = e^{i\tau_k}\,, \tag{A.3}$$

with the orthogonality property

$$\int_0^{2\pi} \frac{d\tau}{2\pi} P^m(\lambda)\bar{P}^l(\lambda) = \delta_{ml}\,. \tag{A.4}$$

Note that in the integral of the product $\Delta \times \bar{\Delta}$ all cross-terms vanish due to the orthogonality of the polynomials, so we are left with $n!$ combinations. In other words, the only surviving permutations are the ones which have for each $P_m(\lambda_i)$ also its complex conjugate.

Now let us consider the integral

$$J_{mn} = \int \prod_{i=1}^{n} \frac{d\tau_i}{2\pi} |\Delta(\tau)|^2 \cos m(\tau_1 - \tau_2) = \int \prod_{i=1}^{n} \frac{d\tau_i}{2\pi} \lambda_1^m \Delta(\tau) \bar{\lambda}_2^m \bar{\Delta}(\tau)\,,$$

and prove the following general result

$$J_{mn}(m > n-1) = 0\,, \qquad J_{mn}(m \le n-1) = -(n-m)(n-2)!\,, \tag{A.5}$$

where we used in the symmetry of $1 \to 2$. First, let us note the following relations

$$\begin{aligned} \lambda_1^m\{\ldots P^l(\lambda_1)\ldots P^{l'}(\lambda_2)\ldots\}(-1)^{\Pi} &= \{\ldots P^{l+m}(\lambda_1)\ldots P^{l'}(\lambda_2)\ldots\}(-1)^{\Pi} \\ \bar{\lambda}_2^m\{\ldots \bar{P}^k(\bar{\lambda}_1)\ldots \bar{P}^{k'}(\bar{\lambda}_2)\ldots\}(-1)^{\Pi'} &= \{\ldots \bar{P}^k(\bar{\lambda}_1)\ldots \bar{P}^{k'+m}(\bar{\lambda}_2)\ldots\}(-1)^{\Pi'}\,. \end{aligned} \tag{A.6}$$

Due to the orthogonality of the polynomials we must have $l+m=k$ and $l'=k'+m$ with all other $n-2$ factors λ_{i_k} distributed the same way in $\Delta(\tau)$ and $\bar{\Delta}(\tau)$. This implies that Π' is an odd permutation of Π and there are $(n-2)!$ ways to distribute λ_{i_k}, $i \neq l, l'$. Finally, the above relations imply that $m+1 \leq l' = l+m \leq n$ and this gives $n-m$ possible values of l. Also, if $m > n-1$ we get vanishing result since $P^{m+l}(\lambda_{i_l})$ cannot be paired with its complex conjugate in $\bar{\Delta}(\tau)$. This leads to (A.5).

One can easily check that (A.5) can be used to reproduce (2.9) and also eq. (28) in [16], i.e.

$$\int \prod_{i=1}^{n} \frac{d\tau_i}{2\pi} |\Delta(\tau)|^2 \left[n + 2\sum_{i<j}^{n} \cos(\tau_i - \tau_j) \right] = n!\,. \tag{A.7}$$

B. Path ordered integral of an antisymmetric function

In this appendix we will show how the path ordered integral $\int [d\tau]_{2n}$ (see (3.6)) of an antisymmetric integrand $A(1,\ldots,2n)$ can be converted into an integral without path-ordering. As is well known, for a symmetric integrand one has

$$\int [d\tau]_{2n} S(1,\ldots,2n) = \frac{1}{(2n)!} \int \prod_{i=1}^{2n} \frac{d\tau_i}{2\pi} S(1,\ldots,2n)\,. \tag{B.1}$$

We can use this identity to show that for a fully antisymmetric integrand one gets

$$\int [d\tau]_{2n} A(1,\ldots,2n) = \frac{1}{(2n)!} \int \prod_{i=1}^{2n} \frac{d\tau_i}{2\pi} \prod_{i<j}^{2n} \epsilon(i,j) A(1,\ldots,2n)\,. \tag{B.2}$$

To prove (B.1) in the symmetric function case we use that $1 = \Theta(i,j) + \Theta(j,i)$ to write

$$\frac{1}{(2n)!} \int \prod_{i=1}^{2n} \frac{d\tau_i}{2\pi} S(1,\ldots,2n) = \frac{1}{(2n)!} \int \prod_{i=1}^{2n} \frac{d\tau_i}{2\pi} \prod_{i<j}^{2n} [\Theta(i,j) + \Theta(j,i)] S(1,\ldots,2n)\,. \tag{B.3}$$

Expanding the product of Theta functions on the r.h.s. we get $2^{\frac{2n(2n-1)}{2}}$ terms out of which only $(2n)!$ have non-circular orderings, i.e. give non-vanishing contributions. Each non-vanishing term has a string $n(2n-1)$ Theta-function factors but only $2n-1$ are needed to get path ordering in the integral over n points. The remaining act as constrains which are automatically satisfied for each of the $(2n)!$ non-vanishing orderings.

Now to prove (3.11) we start from its r.h.s. and that $\epsilon(i,j) = \Theta(i,j) - \Theta(j,i)$. Then going through the same arguments as above we will find all path orderings but with plus or minus sign depending on whether they are odd or even permutations of $\{1,2,\ldots,2n\}$. Using the antisymmetry of $A(1,\ldots,2n)$ we can show that they all are equal to the same path ordered integral $\int \prod_{i=1}^{2n} \frac{d\tau_i}{2\pi} \Theta(1,2)\ldots\Theta(2n-1,2n)$, and that constitutes the proof of (3.11).

C. Properties of $W(1,\ldots,2n)$

Here we shall comment on some properties of the function W defined by (3.8). i.e.

$$W(1,\ldots,2n)=\sum_{\mathcal{P}}(-1)^{\mathcal{P}}\frac{\prod_{i<j}^{2n}D(i,j)}{D(i_1,i_2)\ldots D(i_{2n-1},i_{2n})}\,. \tag{C.1}$$

The summation is over only the $(2n-1)!!$ permutations of pairs of indices with $(-1)^{\mathcal{P}}$ as the symmetry factor of permuting indices between pairs but always in an ordered manner from lower to higher, i.e. $\{(1,2),\ldots,(2n-1,2n)\}\to\{(1,2n),\ldots,(2,2n-1)\}$, etc. If we sum also over exchanges in pairs $(i_1,i_2)\to(i_2,i_1)$ and use the fact that $D(i_1,i_2)=-D(i_2,i_1)$ then we get 2^n extra terms and we need to divide by this factor:

$$W(1,\ldots,2n)=\frac{1}{2^n}\sum_{\mathrm{P}}(-1)^{\mathrm{P}}\frac{\prod_{i<j}^{2n}D(i,j)}{D(i_1,i_2)\ldots D(i_{2n-1},i_{2n})}\,. \tag{C.2}$$

Here P stands for permutations including also interchanges in each pair. There are then $(2n-1)!!\ 2^n$ terms. Next, we can sum over the $n!$ interchanges of pairs among themselves. This way we can write $W(1,\ldots,2n)$ as a sum over all $(2n)!$ permutations. The final form we get is:

$$W(1,\ldots,2n)=\frac{1}{2^n n!}\sum_{\Pi}(-1)^{\Pi}\frac{\prod_{i<j}^{2n}D(i,j)}{D(i_1,i_2)\ldots D(i_{2n-1},i_{2n})}\,, \tag{C.3}$$

where Π stands for all permutations.

D. Details of integral evaluation in section 3.1

To compute the superstring partition function to zero order in expansion in derivatives (3.12) one needs to find the constant (non-oscillating) part of $W(1,\ldots,2n)$. We can rewrite the integrand in (3.12) as:

$$\begin{aligned}W(1,\ldots,2n)\ &=\sum_{\mathcal{P}}(-1)^{\mathcal{P}}(\bar{P}^n(i_1)\bar{P}^{n-1}(i_2)\ldots\bar{P}^n(i_{2n-1})\bar{P}^{n-1}(i_{2n}))\times\\&\times\frac{\Delta(\tau)}{(1-\bar{P}(i_1)P(i_2))\ldots(1-\bar{P}(i_{2n-1})P(i_{2n}))}\,.\end{aligned} \tag{D.1}$$

Expanding the denominators in power series we get

$$\begin{aligned}W(1,2,\ldots,2n)\ &=\sum_{\mathcal{P}}(-1)^{\mathcal{P}}(\bar{P}^n(i_1)\bar{P}^{n-1}(i_2)\ldots\bar{P}^n(i_{2n-1})\bar{P}^{n-1}(i_{2n}))\times\\&\times\left[\Delta(\tau)\prod_{k=1}^{n}\sum_{m_k}(\bar{P}^{m_k}(i_{2k-1})P^{m_k}(i_{2k}))\right].\end{aligned} \tag{D.2}$$

From the second term (in square brackets) we need to pick up the complex conjugate to the first one.

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Let us first consider the $\{1,2,\ldots,2n\}$ ordering in the sum over permutations. The basic observations which we use in our computation are: (a) in $\Delta(\tau)$ each ordering of powers of the polynomials appears only once and each polynomial $P(i_k)$ has a different power compared to others; (b) if from $\prod_{k=1}^{n}\sum_{m_k}\ldots$ we pick up the term $\bar{P}(i_1)^s P(i_2)^s$ $(m_1=s)$ then from $\Delta(\tau)$ we must pick up the term $P(i_1)^{n+s}P(i_2)^{n-s-1}$ with $n+s\le 2n-1$, $0\le n-1-s$; (c) since each power $0,\ldots,2n-1$ appears in $\Delta(\tau)$ only once, all possible powers $m_k=0,\ldots,n-1$ are different from each other; (d) there are $n!$ permutations distributing the distinct m_k powers over the pairs of polynomials that appear in (D.2) and we need only to determine the relative signs of them.

Let $S^0 = P^{2n-1}(1)P^{2n-2}(2)\ldots P^0(2n)$ be a reference configuration in $\Delta(\tau)$ with $(-1)^{\Pi(S^0)}$ =+1. If we want construct a configuration with a given distribution of powers $\{m_k\}$ then we proceed from the S^0 as follows: (a) Permute the pair of indices (i_{2k-1},i_{2k}) which has the maximum $m_k = n-1$ to the position such that $P(i_{2k-1})$ has power $n+m_k = 2n-1$ and $P(i_{2k})$ has power $2n-2$. For this step we need an even number of permutation since we are permuting a pair. (b) Now take the i_{2k} index $2m_k = 2n-2$ positions to the right to power zero. (c) Follow the same procedure for all other pairs of indices in descending order of powers. In this way we can construct all strings of polynomials $P(i_1)^{2n-1}P(i_2)^{2n-2}\ldots P(i_{2n})^0$ by making an even number of permutations. We conclude that all the terms in $\Delta(\tau)$ which are picked up by this procedure have $(-1)^{\Pi(S^{\{m_k\}})} = (-1)^{\Pi(S^0)} = +1$. Therefore, they all add up to give a contribution of the term with $(1,2,\ldots,2n)$ ordering in $W(1,\ldots,2n)$ to be $n!$.

The final step is to find the contribution of the remaining $\mathcal{P}\{1,\ldots,2n\}$ terms in W. This is easily done if we notice that we can write $\Delta(\tau)=\prod_{i<j}(P(i)-P(j)) = (-1)^{\mathcal{P}\{1,\ldots,2n\}}$ $\prod_{\mathcal{P}(i<j)}(P(i)-P(j))$. Here $\mathcal{P}(i<j)$ means the permutation of the initial ordering, i.e. if $\mathcal{P}(1234\ldots) = (1324\ldots)$ then $\mathcal{P}(i<j) = \mathcal{P}(1<2<3<4\cdots) = (1<3<2<4\cdots)$. We then proceed in the same way with the proof as above to find that the contribution is equal to $(-1)^{\mathcal{P}}n!$. The final result is then:

$$\int\prod_{i=1}^{2n}\frac{d\tau_i}{2\pi}W(1,\ldots,2n) = \sum_{\mathcal{P}}(-1)^{\mathcal{P}}[(-1)^{\mathcal{P}}n!] = (2n-1)!!n!\,. \tag{D.3}$$

E. Details of integral evaluation in section 3.2

In this appendix we will discuss computation of the integral C_n from (3.21). First, we write the cosine as $P(2)^M\bar{P}(1)^M$ where we used the symmetry of the integrand under $1\to 2$. In the definition of $W(1,\ldots,2n)$ there are two kinds of terms: (i) those where $D(1,2)$ appears in the denominator of the fractions in (3.12) and (ii) those where $D(1,*)D(2,*)$ appears instead, where $*$ is any other index from $\{3,4,\ldots\}$.

The first case is easily worked out. Expanding the denominators we get:

$$\begin{aligned}&(-1)^{\mathcal{P}(12\ldots)}\bar{P}(1)^M P(2)^M(\bar{P}^n(i_1)\bar{P}^{n-1}(i_2)\ldots\bar{P}^n(i_{2n-1})\bar{P}^{n-1}(i_{2n}))\times\\ &\times\Delta(\tau)\prod_{k=1}^{n}\sum_{m_k}(\bar{P}^{m_k}(i_{2k-1})P^{m_k}(i_{2k}))\,.\end{aligned}\tag{E.1}$$

As in the computation of $\mathcal{Z}_0$ the integral will project from $\Delta(\tau)$ those strings of polynomials which are conjugates to the rest of the integrand for each permutation of powers $\{m_k\} \equiv \{0,\ldots,n-1\}$. There are two constrains imposed by the minimal (0) and maximal $(2n-1)$ power in $\Delta(\tau)$: $1 \leq M \leq N-1$ and $0 \leq m_1 \leq n-1-M$. Therefore, there are exactly $(n-M)$ possible values of m_1 and the remaining $n-1$ powers m_k can be distributed in $(n-1)!$ ways over the remaining pairs of indices. Each of the $(n-M)\times(n-1)!$ terms has the same sign, $(-1)^{\mathcal{P}\{12\ldots\}}$, since the required strings of polynomials in $\Delta(\tau)$ are constructed just like in the zeroth order calculation in appendix D. The total contribution of these terms is found after summing over all $(2n-3)!!$ combinations in $W(1,\ldots,2n)$ which have $D(1,2)$ in the denominator:

$$\begin{aligned} Q_n^{(1)}(M) &= \sum_{\{(12),\ldots\}} (n-M)\times(n-1)![(-1)^{\mathcal{P}(1,2,\ldots,2n)}]^2 \\ &= (2n-3)!!(n-M)(n-1)!\,. \end{aligned} \tag{E.2}$$

The second type of terms corresponds to the remaining $(2n-2)(2n-3)!!$ ways of pairing indices 1 and 2 with any other index except with each other. Here a typical term is:

$$\begin{aligned} &(-1)^{\mathcal{P}(1,i_3,2,i_4\ldots)}\bar{P}(1)^M P(2)^M(\bar{P}^n(i_1)\bar{P}^{n-1}(i_3)\bar{P}^n(i_2)\bar{P}^{n-1}(i_4)\ldots\Delta(\tau)\times \\ &\quad\times\sum_{m_1}(\bar{P}(1)^{m_1}P(3)^{m_1})\sum_{m_2}(\bar{P}(2)^{m_1}P(4)^{m_2})\prod_{k=3}^{n}\sum_{m_k}(\bar{P}^{m_k}(i_{2k-1})P^{m_k}(i_{2k}))\,. \end{aligned} \tag{E.3}$$

The determinant imposes again the constrains: $1 \leq M \leq N-1$ and $0 \leq m_{1,2} \leq n-1-M$. The remaining $n-2$ powers m_k are distributed between the remaining $n-2$ pairs. It is obvious from the analysis of appendix D, that the two members of a given pair of polynomials are split symmetrically with respect to the pair with powers $(n, n-1)$, i.e. $P(i_{2k-1})^{n+m_k}\cdots P(i_{2k'-1})^n P(i_{2k'})^{n-1}\ \ldots P(i_{2k})^{n-1-m_k}$. In our case this implies that we must have $m_2 = m_1 + M$. This means that the pairs $(1, i_4)$ and $(2, i_3)$ are the ones with symmetric polynomial powers above and below the pair of polynomials with powers $(n, n-1)$ respectively. We would like to follow the method of appendix D to construct the required strings of polynomials from the reference one, S^0. As a first step we have to permute $(1, i_3, 2, i_4, \ldots) \to (1, i_4, 2, i_3, \ldots)$ in $\Delta(\tau)$ resulting in a minus sign. The rest goes as in the first case with all terms in each $\mathcal{P}\{1, i_3, 2, i_4\ldots\}$ contributing the same quantity $-(-1)^{\mathcal{P}\{1,i_3,2,i_4\ldots\}}(n-M)(n-2)!$ and finally:

$$Q_n^{(2)}(M) = -(2n-2)(2n-3)!!(n-M)(n-2)!\,. \tag{E.4}$$

Adding the two expression in (E.2) and (E.4) for each M leads to the result in (3.22).

Let us consider an explicit example which will make the procedure more transparent. Take the case $n = 3$ and let us look at the following term of the type (ii):

$$\begin{aligned} &(-1)^{\mathcal{P}(1,3,2,4,5,6)}\bar{P}(1)^M P(2)^M(\bar{P}^3(1)\bar{P}^2(3)\bar{P}^3(2)\bar{P}^2(4)\bar{P}^3(5)\bar{P}^2(6)\times \\ &\quad\times\Delta(\tau)(\bar{P}(1)^{m_1}P(3)^{m_1})(\bar{P}(2)^{m_1}P(4)^{m_2})(\bar{P}^{m_3}(5)P^{m_3}(6))\,, \end{aligned} \tag{E.5}$$

where we have suppressed the m_1, m_2, m_3, M summations. Here the powers x_i of each polynomial $P^{x_i}(i)$ are:

$$(x_1, x_3, x_2, x_4, x_5, x_6) = (3+M+m_1, 2-m_1, 3+m_2-M, 2-m_2, 3+m_3, 2-m_3)\,,$$

where we used the identity $P(i)^m = \bar{P}(i)^{-m}$. Since in $\Delta(\tau)$ there are only polynomials with degrees 0 to 5, this implies that $1 \leq M \leq 2$ and $0 \leq m_{1,2} \leq 2 - M$. Moreover, $0 \leq m_3 \leq 2$. We see that the last pair $(5, 6)$ has powers symmetrically m_3 steps higher and m_3 steps lower with respect to the pair which has powers $(3, 2)$. Also, from the powers of $(1, 3, 2, 4)$ we conclude that two of them (those of $P(1), P(2)$) are 3 or higher and the other two (of $P(3), P(4)$) are 2 or lower.

Now we have two possibilities: (i) $P(1)$ and $P(3)$ have powers symmetric with respect to the center (and the same for $P(2)$ and $P(4)$) implying $3 + M + m_1 = 3 + m_1$ which is impossible since $M \geq 1$, and (ii) $P(1)$ and $P(2)$ have powers symmetric with respect to the center (and the same for $P(2)$ and $P(3)$) which implies $3 + M + m_1 = 3 + m_2$. Then our string of polynomials becomes:

$$\bar{P}(1)^{3+M+m_1}\,\bar{P}(3)^{2-m_1}\,\bar{P}(2)^{3+m_1}\,\bar{P}(4)^{2-M-m_1}\,\bar{P}(5)^{3+m_3}\,\bar{P}(6)^{2-m_3}\,.$$

Now there are two possibilities: $M = 1$ and one for $M = 2$ which lead to distinct powers $\bar{x}_i = -x_i$ of each of the six polynomials $\bar{P}^{\bar{x}_i}$ (i)

M	m_1	m_3	$(\bar{x}_1, \bar{x}_4, \bar{x}_2, \bar{x}_3, \bar{x}_5, \bar{x}_6)$	$(-1)^{\Pi(S^{\{m_k\}})}$
1	0	2	$(4, 1, 3, 2, 5, 0)$	$+1$
1	1	0	$(5, 0, 4, 1, 3, 2)$	$+1$
2	0	1	$(5, 0, 3, 2, 4, 1)$	$+1$

(E.6)

In the last column we have given the sign of each string of polynomials in

$$\bar{\Delta}'(\tau) = (-1)^{\mathcal{P}\{1,4,2,3,5,6\}} \prod_{\mathcal{P}(i<j)} (\bar{P}(i) - \bar{P}(j))$$

with reference to the

$$S' = \bar{P}(1)^5 \bar{P}(4)^4 \bar{P}(2)^3 \bar{P}(3)^2 \bar{P}(5)^1 \bar{P}(6)^0$$

string. The signs of the strings of polynomials of the fourth column in the table are all positive relative to S'. One can show this by using the method of appendix (D). We have to permute the indices of the reference string S' an even number of times to construct any of the strings in the fourth column of the table, e.g., for the third entry in the table above:

$$\begin{aligned}\bar{P}(1)^5 \bar{P}(4)^4 \bar{P}(2)^3 \bar{P}(3)^2 \bar{P}(5)^1 \bar{P}(6)^0 &\to \bar{P}(1)^5 \bar{P}(2)^4 \bar{P}(3)^3 \bar{P}(5)^2 \bar{P}(6)^1 \bar{P}(4)^0 \\ &\to \bar{P}(1)^5 \bar{P}(5)^4 \bar{P}(6)^3 \bar{P}(2)^2 \bar{P}(3)^1 \bar{P}(4)^0 \\ &\to \bar{P}(1)^5 \bar{P}(5)^4 \bar{P}(2)^3 \bar{P}(3)^2 \bar{P}(6)^1 \bar{P}(4)^0\,.\end{aligned}$$

This is the desired result, and if we use the fact that $\Delta(\tau) = (-1)^{\mathcal{P}\{1,4,2,3,5,6\}}\Delta'(\tau)$ in (E.5) we get an overall minus sign. There are 12 such cases as the one we studied above. Therefore, $Q_3^{(2)}(1) = -12 \times 2 = -24$ and $Q_3^{(2)}(2) = -12 \times 1 = -12$, in agreement with the general expression (E.4).

F. Details of computation of stress-energy tensor

The computation of the stress-energy tensor uses the same technology as developed in the previous appendices. The computation of the M_{2n} integral (4.20) in section 4 is quite lengthy and here we shall give a short account of some intermediate results.

The integral we are to compute is

$$M_{2n} = -\frac{1}{(2n)!}\int \prod_{l=1}^{2n} \frac{d\tau_l}{2\pi} W(1,2,\ldots 2n) \sum_{i<j} \ln|e^{i\tau_i} - e^{i\tau_j}|^2 \sum_{i<j} \cos(\tau_i - \tau_j)\,. \tag{F.1}$$

Due to the symmetry under interchange of the integrations points it can be written as:

$$\begin{aligned} M_{2n} &= -\frac{1}{(2n)!}\int \prod_{i=1}^{2n} \frac{d\tau_l}{2\pi} W(1,2,\ldots,2n) \sum_{i<j} \ln|e^{i\tau_1} - e^{i\tau_2}| \times \\ &\quad \times 2[\cos(\tau_1-\tau_2) + 2(2n-2)\cos(\tau_1-\tau_3) + (n-1)(2n-3)\cos(\tau_3-\tau_4)]\,. \end{aligned} \tag{F.2}$$

In order to compute the integral for each of the three terms in the brackets we expand the logarithm in a series of cosines.

Then for each term we will need to distinguish terms in $W(1,2\ldots 2n)$ (C.1) depending on how the special points $1,2,3,4$ of the logarithm and $\cos(\tau_i - \tau_j)$ appear in the $D(ij)$ of the denominator. In the table that follows we present the results for each type of terms. The first column contains the structure of the denominator for the terms selected. The second column gives the product of orthogonal polynomials selected from the cosine function. The third column gives the number of independent configurations of a given type in W. The fourth column contains the result of integration computed in the same way as the derivative corrections in the partition function.

In the expansion of the logarithm we use the symmetries of the rest of the integrand to write: $\ln|e^{i\tau_1} - e^{i\tau_2}| = \sum_{M=0}^{\infty} \frac{\cos[(M+1)(t_1-t_2)]}{M+1} = \sum_{M=0}^{\infty} \frac{\bar{P}(1)^{M+1}P(2)^{M+1}}{M+1}$. Summation over the integer variable M is always implied. Also, we use the convention $D(ij) = D(i,j)$ and $\Theta(k) = \Theta(n-k) = 1$, if $n > k$ or $= 0$ if $n < k$.

Combining the results given in the tables below we get the final expression (4.20) for the integral in (F.2).

I: $\ln|e^{i\tau_1} - e^{i\tau_2}|\cos(\tau_1 - \tau_2)$**:**

Denominator	Polynomials	Combinations	Integral
$D(12)$	$\bar{P}(1)P(2)$	$(2n-3)!!$	$-\frac{1}{2}(n-1)!(n-2-M)\Theta(n-3-M)\Theta(3)$
$D(1*)D(2*)$	$\bar{P}(1)P(2)$	$(2n-2)(2n-3)!!$	$+\frac{1}{2}(n-2)!(n-2-M)\Theta(n-3-M)\Theta(3)$
$D(12)$	$\bar{P}(2)P(1)$	$(2n-3)!!$	$-\frac{1}{2}((n-1)!(n-M)\Theta(n-1-M)\Theta(2) - \delta_{n,1}\delta_{M,0})$
$D(1*)D(2*)$	$\bar{P}(2)P(1)$	$(2n-2)(2n-3)!!$	$\frac{1}{2}(-n^2(n-2)!\Theta(2)\delta_{M,0}$ $+(n-2)!(n-M)\Theta(n-1-M)\Theta(2))$

(F.3)

II: $\ln|e^{i\tau_1} - e^{i\tau_2}|\cos(\tau_1 - \tau_3)$**:**

Denominator	Polynomials	Combinations	Integral
$D(13)D(2*)$	$\bar{P}(1)P(3)$	$(2n-3)!!$	$+\frac{1}{4}(n-2)!(n-2-M)\Theta(n-3-M)\Theta(3)$
$D(13)D(2*)$	$\bar{P}(3)P(1)$	$(2n-3)!!$	$+\frac{1}{4}(n-2)!(n-1-M)\Theta(n-2-M)\Theta(2)$
$D(13)D(2*)$	$\bar{P}(2)P(3)$	$(2n-3)!!$	$+\frac{1}{4}(n-2)!(n-2-M)\Theta(n-3-M)\Theta(3)$
$D(13)D(2*)$	$\bar{P}(3)P(2)$	$(2n-3)!!$	$+\frac{1}{4}((n-2)!(n-1-M)\Theta(n-2-M)\|_{M\neq 0}\Theta(2)$ $-(n-1)^2\delta_{M,0}\Theta(2))$
$D(23)D(1*)$	$\bar{P}(1)P(3)$	$(2n-3)!!$	$+\frac{1}{4}(n-2)!(n-2-M)\Theta(n-3-M)\Theta(3)$
$D(23)D(1*)$	$\bar{P}(3)P(1)$	$(2n-3)!!$	$+\frac{1}{4}((n-2)!(n-1-M)\Theta(n-2-M)\|_{M\neq 0}\Theta(2)$ $-(n-1)^2\delta_{M,0}\Theta(2))$
$D(23)D(1*)$	$\bar{P}(2)P(3)$	$(2n-3)!!$	$+\frac{1}{4}(n-2)!(n-1-M)\Theta(n-2-M)\Theta(2)$
$D(23)D(1*)$	$\bar{P}(3)P(2)$	$(2n-3)!!$	$+\frac{1}{4}(n-2)!(n-2-M)\Theta(n-3-M)\Theta(3)$
$D(12)D(3*)$	$\bar{P}(1)P(3)$	$(2n-3)!!$	$+\frac{1}{4}(n-2)!(n-2-M)\Theta(n-3-M)\Theta(3)$
$D(12)D(3*)$	$\bar{P}(3)P(1)$	$(2n-3)!!$	$+\frac{1}{4}(n-2)!(n-1-M)\Theta(n-2-M)\Theta(2)$
$D(12)D(3*)$	$\bar{P}(2)P(3)$	$(2n-3)!!$	$+\frac{1}{4}(n-2)!(n-1-M)\Theta(n-2-M)\Theta(2)$
$D(12)D(3*)$	$\bar{P}(3)P(2)$	$(2n-3)!!$	$+\frac{1}{4}(n-2)!(n-2-M)\Theta(n-3-M)\Theta(3)$
$D(1*)D(2*)D(3*)$	$\bar{P}(1)P(3)$	$(2n-4)(2n-3)!!$	$-\frac{1}{4}2(n-3)!(n-2-M)\Theta(n-3-M)\Theta(3)$
$D(1*)D(2*)D(3*)$	$\bar{P}(3)P(1)$	$(2n-4)(2n-3)!!$	$-\frac{1}{4}(2(n-2)!(n-1-M)\Theta(n-2-M)\|_{M\neq 0}\Theta(3)$ $-(n-1)!\delta_{M,0}\Theta(3))$
$D(1*)D(2*)D(3*)$	$\bar{P}(2)P(3)$	$(2n-4)(2n-3)!!$	$-\frac{1}{4}(2(n-2)!(n-1-M)\Theta(n-2-M)\|_{M\neq 0}\Theta(3)$ $-(n-1)!\delta_{M,0}\Theta(3))$
$D(1*)D(2*)D(3*)$	$\bar{P}(3)P(2)$	$(2n-4)(2n-3)!!$	$-\frac{1}{4}2(n-2)!(n-2-M)\Theta(n-3-M)\Theta(3)$

(F.4)

III: $\ln|e^{i\tau_1} - e^{i\tau_2}|\cos(\tau_3 - \tau_4)$**:**

Denominator	Polynomials	Combinations	Integral
$D(12)D(34)$	$\bar{P}(3)P(4)$	$(2n-5)!!$	$-(n-2)(n-2)!(n-1-M)\Theta(n-2-M)\Theta(2)$
$D(13)D(24)$	$\bar{P}(3)P(4)$	$(2n-5)!!$	$+(n-2)!(n-2-M)\Theta(n-3-M)\Theta(3)$
$D(14)D(23)$	$\bar{P}(3)P(4)$	$(2n-5)!!$	$+(n-2)!((n-1-M)\Theta(n-2-M)\Theta(2)$ $-(n-1)^2\delta_{M,0}\Theta(2))$
$D(1*)D(2*)D(34)$	$\bar{P}(3)P(4)$	$(2n-4)(2n-5)!!$	$+(n-3)!((n-2)(n-1-M)\Theta(n-2-M)\Theta(3)$ $(n-2-M)\Theta(n-3-M)\Theta(3))$
$D(3*)D(4*)D(12)$	$\bar{P}(3)P(4)$	$(2n-4)(2n-5)!!$	$+(n-3)!((n-2)(n-1-M)\Theta(n-2-M)\Theta(3)$ $(n-2-M)\Theta(n-3-M)\Theta(3))$
$D(1*)D(3*)D(24)$	$\bar{P}(3)P(4)$	$(2n-4)(2n-5)!!$	$-2(n-3)!(n-2-M)\Theta(n-3-M)\Theta(3)$
$D(3*)D(4*)D(12)$	$\bar{P}(3)P(4)$	$(2n-4)(2n-5)!!$	$-2(n-3)!(n-2-M)\Theta(n-3-M)\Theta(3)$
$D(1*)D(4*)D(23)$	$\bar{P}(3)P(4)$	$(2n-4)(2n-5)!!$	$-(n-3)!((n-1-M)\Theta(n-2-M)$ $+(n-2-M)\Theta(n-3-M)-(n-1)^2\delta_{M,0})\Theta(3)$
$D(3*)D(4*)D(12)$	$\bar{P}(3)P(4)$	$(2n-4)(2n-5)!!$	$-(n-3)!((n-1-M)\Theta(n-2-M)$ $+(n-2-M)\Theta(n-3-M)-(n-1)^2\delta_{M,0})\Theta(3)$
$D(1*)D(2*)\times$ $D(3*)D(4*)$	$\bar{P}(3)P(4)$	$(2n-4)(2n-6)\times$ $(2n-5)!!$	$-(n-4)!((n-3)(n-1-M)\Theta(n-2-M)$ $-4(n-2-M)\Theta(n-3-M)+(n-1)^2\delta_{M,0})\Theta(4)$

(F.5)

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